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# ON THE FEKETE-SZEGÖ PROBLEM FOR STRONGLY $\alpha$ -LOGARITHMIC QUASICONVEX FUNCTIONS

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**ABSTRACT.** The purpose of the present paper is to introduce the classes  $\mathcal{M}^\alpha(\beta)$  and  $\mathcal{Q}^\alpha(\beta)$ , respectively, of normalized strongly  $\alpha$ -logarithmic convex and quasiconvex functions of order  $\beta$  in the open unit disk and to obtain sharp Fekete-Szegő inequalities for functions belonging to the classes be the class  $\mathcal{M}^\alpha(\beta)$  and  $\mathcal{Q}^\alpha(\beta)$ .

## 1. Introduction

Let  $\mathcal{S}$  denote the class of analytic functions  $f$  of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are univalent in the open unit disk  $\mathcal{U} = \{z : |z| < 1\}$ . A classical theorem of Fekete and Szegő [8] states that for  $f \in \mathcal{S}$  given by (1.1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0, \\ 1 + 2e^{-2\mu/(1-\mu)} & \text{if } 0 \leq \mu \leq 1, \\ 4\mu - 3 & \text{if } \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each  $\mu$  there exists a function in  $\mathcal{S}$  such that equality holds. Recently, Pfluger [17,18] has considered the problem when  $\mu$  is complex. In the case of  $\mathcal{C}$ ,  $\mathcal{S}^*$  and  $\mathcal{K}$ , the subclasses of convex, starlike and close-to-convex functions, respectively, the above inequality can be improved [10,11]. Also, Darus and Thomas [5] studied the class  $\mathcal{M}^\alpha$  of  $\alpha$ -logarithmic convex functions and they also have solved the Fekete-Szegő problem for the class  $\mathcal{M}^\alpha$ . Furthermore, London [14] have extended the results of Abdel-Gawad and Thomas [1], Keogh and

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Merkes [10] and Koepf [11,12] to the class  $\mathcal{K}(\beta)$  of strongly close-to-convex functions of order  $\beta$ . Now we introduce new classes which incorporate well-known classes of univalent functions.

**Definition 1.1.** A function  $f \in \mathcal{S}$  given by (1.1) is said to be strongly logarithmic  $\alpha$ -convex of order  $\beta$  if

$$\left| \arg \left\{ \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{f'(z)} \right)^\alpha \right\} \right| \leq \frac{\pi}{2} \beta \quad (\alpha \geq 0; 0 < \beta \leq 1; z \in \mathcal{U}). \quad (1.2)$$

Denote by  $\mathcal{M}^\alpha(\beta)$  the class of strongly  $\alpha$ -logarithmic convex functions of order  $\beta$ . The class  $\mathcal{M}^\alpha(\beta)$  was introduced by Chiang [4]. In particular, the classes  $\mathcal{M}^\alpha(1) = \mathcal{M}^\alpha$  and  $\mathcal{M}^0(\beta)$  have been extensively studied by Lewandowski, Miller and Zlotkiewicz [13] and Bramnan and Kirwan [2](also, see [7,20]), respectively.

**Definition 1.2.** A function  $f \in \mathcal{S}$  given by (1.1) is said to be  $\alpha$ -logarithmic quasiconvex of order  $\beta$  if there exists a function  $g \in \mathcal{C}$  such that

$$\left| \arg \left\{ \left( \frac{f'(z)}{g'(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{g'(z)} \right)^\alpha \right\} \right| \leq \frac{\pi}{2} \beta \quad (\alpha, \beta \geq 0; z \in \mathcal{U}). \quad (1.3)$$

We denote by  $\mathcal{Q}^\alpha(\beta)$  the class of strongly  $\alpha$ -logarithmic quasiconvex functions of order  $\beta$ . Clearly,  $\mathcal{Q}^0(1)$  and  $\mathcal{Q}^1(1)$  are the classes of close-to-convex functions and quasiconvex functions introduced by Kaplan [9] and Noor [15](also, see [16]), respectively. Also we note that  $\mathcal{Q}^0(\beta) = \mathcal{K}(\beta)$ .

In the present paper, we derive sharp Fekete-Szegő inequalities for functions belonging to the classes  $\mathcal{M}^\alpha(\beta)$  and  $\mathcal{Q}^\alpha(\beta)$ , which imply the results obtained by Abdel-Gawad and Thomas [1], Darus and Thomas [5], Keogh and Merkes [10], Koepf [11,12], and London [14].

## 2. Results

To prove our main results, we need the following

**Lemma 2.1.** Let  $p$  be analytic in  $\mathcal{U}$  and satisfy  $\operatorname{Re} \{p(z)\} > 0$  for  $z \in \mathcal{U}$ , with  $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ . Then

$$|p_n| \leq 2 \quad (n \geq 1) \quad (2.1)$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \quad (2.2)$$

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The inequality (2.1) was first proved by Carathéodory [3] (also, see Duren [6, p. 41]) and the inequality (2.2) can be found in [19, p.166].

With the help of Lemma 2.1, we now derive

**Theorem 2.1.** *Let  $f \in \mathcal{M}^\alpha(\beta)$  and be given by (1.1). Then for complex number  $\mu$ ,*

$$|a_3 - \mu a_2^2| \leq \frac{\beta}{1+2\alpha} \max \left\{ 1, \frac{|3(1+3\alpha) - 4\mu(1+2\alpha)|\beta}{(1+\alpha)^2} \right\}.$$

*For each  $\mu$ , there is a function in  $\mathcal{M}^\alpha(\beta)$  such that equality holds.*

*Proof.* From (1.2), we can write

$$\left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))'}{f'(z)} \right)^\alpha = p^\beta(z),$$

where  $p$  is given by Lemma 2.1. Equating coefficients, we obtain

$$a_2 = \frac{\beta}{1+\alpha} p_1 \quad (2.3)$$

and

$$a_3 = \frac{1}{4(1+2\alpha)} \left( \beta(\beta-1)p_1^2 + 2\beta p_2 - (\alpha^2 - 7\alpha - 2) \left( \frac{\beta p_1}{1+\alpha} \right)^2 \right).$$

Then we have

$$a_3 - \mu a_2^2 = \frac{\beta}{2(1+2\alpha)} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{(3+9\alpha-4\mu(1+2\alpha))\beta^2 p_1^2}{4(1+2\alpha)(1+\alpha)^2}. \quad (2.4)$$

Hence (2.4) and Lemma 2.1 give

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta}{2(1+2\alpha)} \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{|3+9\alpha-4\mu(1+2\alpha)|\beta^2 |p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \\ &\leq \frac{\beta}{1+2\alpha} + \frac{\{|3+9\alpha-4\mu(1+2\alpha)|\beta^2 - (1+\alpha)^2\beta\} |p_1|^2}{4(1+2\alpha)(1+\alpha)^2}. \end{aligned}$$

Therefore, by using  $|p_1| \leq 2$ , we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } k(\alpha) \leq \frac{(1+\alpha)^2}{\beta}, \\ \frac{|3+9\alpha-4\mu(1+2\alpha)|\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } k(\alpha) \geq \frac{(1+\alpha)^2}{\beta}, \end{cases}$$

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where

$$k(\alpha) = |3(1 + 3\alpha) - 4\mu(1 + 2\alpha)|.$$

Equality is attained for functions in  $\mathcal{M}^\alpha(\beta)$ , respectively, given by

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \left(\frac{(zf'(z))'}{f'(z)}\right)^\alpha = \left(\frac{1+z^2}{1-z^2}\right)^\beta \quad (2.5)$$

and

$$\left(\frac{zf'(z)}{f(z)}\right)^{1-\alpha} \left(\frac{(zf'(z))'}{f'(z)}\right)^\alpha = \left(\frac{1+z}{1-z}\right)^\beta. \quad (2.6)$$

**Remark 2.1.** It follows at once from (2.3) that  $|a_2| \leq 2\beta/(1 + \alpha)$  and Theorem 2.1 gives

$$|a_3| \leq \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } (1 + \alpha)^2 \geq 3(1 + 3\alpha)\beta, \\ \frac{3(1+3\alpha)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } (1 + \alpha)^2 \leq 3(1 + 3\alpha)\beta, \end{cases}$$

The inequality for  $|a_2|$  is sharp when  $f$  is defined by (2.6) and the inequalities for  $|a_3|$  are sharp when  $f$  is defined by (2.5) and (2.6), respectively.

Next, we consider the real number  $\mu$  as follows.

**Theorem 2.2.** Let  $f \in \mathcal{M}^\alpha(\beta)$  and be given by (1.1). Then for real number  $\mu$ ,

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(3(1+3\alpha) - 4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \leq \frac{3(1+3\alpha)\beta - (1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)\beta - (1+\alpha)^2}{4(1+2\alpha)\beta} \leq \mu \leq \frac{3(1+3\alpha)\beta + (1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{(4(1+2\alpha)\mu - 3(1+3\alpha))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \geq \frac{3(1+3\alpha)\beta + (1+\alpha)^2}{4(1+2\alpha)\beta}. \end{cases}$$

For each  $\mu$ , there is a function in  $\mathcal{C}^\alpha(\beta)$  such that equality holds in all cases.

*Proof.* We consider two cases. At first, we suppose that  $\mu \leq 3(1+3\alpha)/(4(1+2\alpha))$ . Then (2.3) and Lemma 2.1 give

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta}{2(1+2\alpha)} \left(2 - \frac{|p_1|^2}{2}\right) + \frac{(3 + 9\alpha - 4\mu(1+2\alpha))\beta^2 |p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \\ &\leq \frac{\beta}{1+2\alpha} + \frac{((3 + 9\alpha - 4\mu(1+2\alpha))\beta^2 - (1+\alpha)^2\beta) |p_1|^2}{4(1+2\alpha)(1+\alpha)^2}. \end{aligned}$$

So, by using the fact that  $|p_1| \leq 2$ , we obtain

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$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(3(1+3\alpha)-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \leq \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta} \leq \mu \leq \frac{3(1+3\alpha)}{4(1+2\alpha)}, \end{cases}$$

Equality is attained by choosing  $p_1 = p_2 = 2$  and  $p_1 = 0, p_2 = 2$ , respectively, in (2.3).

Next, we suppose that  $\mu \geq 3(1+3\alpha)/(4(1+2\alpha))$ . In this case, it follows again from (2.3) and Lemma 2.1 that

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{\beta}{2(1+2\alpha)} \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{(4\mu(1+2\alpha) - (3+9\alpha))\beta^2 |p_1|^2}{4(1+2\alpha)(1+\alpha)^2} \\ &\leq \frac{\beta}{1+2\alpha} + \frac{((4\mu(1+2\alpha) - (3+9\alpha))\beta^2 - \beta(1+\alpha)^2) |p_1|^2}{4(1+2\alpha)(1+\alpha)^2}, \end{aligned}$$

and so, as in the first case, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\beta}{1+2\alpha}, & \text{if } \frac{3(1+3\alpha)}{4(1+2\alpha)} \leq \mu \leq \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}, \\ \frac{(4(1+2\alpha)\mu-3(1+3\alpha))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if } \mu \geq \frac{3(1+3\alpha)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}. \end{cases}$$

The results are sharp by choosing  $p_1 = 0, p_2 = 2$  and  $p_1 = 2i, p_2 = -2$ , respectively, in (2.3).

**Remark 2.2.** If we take  $\beta = 1$  in Theorem 2.1 and Theorem 2.2, then we obtain the results by Darus and Thomas [5].

Finally, we prove

**Theorem 2.3.** Let  $f \in Q^\alpha(\beta)$  and be given by (1.1). Then for  $\alpha \geq 0$  and  $\beta \geq 0$ , we have

$$3(2\alpha+1) |a_3 - \mu a_2^2| \leq \begin{cases} 1 + \frac{(1+\beta)^2(2(3\alpha+1)-3(2\alpha+1)\mu)}{(\alpha+1)^2}, & \text{if } \mu \leq \frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)}, \\ 1 + 2\beta + \frac{2(2(3\alpha+1)-3(2\alpha+1)\mu)}{2(\alpha+1)^2 - \beta(2(3\alpha+1)-3(2\alpha+1)\mu)}, & \text{if } \frac{2\alpha(1-\alpha)+2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)} \leq \mu \leq \frac{2(3\alpha+1)}{3(2\alpha+1)}, \\ 1 + 2\beta, & \text{if } \frac{2(3\alpha+1)}{3(2\alpha+1)} \leq \mu \leq \frac{2(\alpha+1)^2+2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}, \\ -1 + \frac{(1+\beta)^2(3(2\alpha+1)\mu-2(3\alpha+1))}{(\alpha+1)^2}, & \text{if } \mu \geq \frac{2(\alpha+1)^2+2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}. \end{cases}$$

For each  $\mu$ , there is a function in  $Q^\alpha(\beta)$  such that equality holds in all cases.

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*Proof.* Let  $f \in Q^\alpha(\beta)$ . Then it follows from (1.2) that we may write

$$\left(\frac{f'(z)}{g'(z)}\right)^{1-\alpha} \left(\frac{(zf'(z))'}{g'(z)}\right)^\alpha = p^\beta(z), \quad (2.7)$$

where  $g$  is convex and  $p$  has positive real part. Let  $g(z) = z + b_2z^2 + b_3z^3 + \dots$  and let  $p$  be given in the Lemma above. Then by comparing the coefficients of both sides of (2.7), we obtain

$$2(\alpha + 1)a_2 = \beta p_1 + 2b_2$$

and

$$\begin{aligned} 3(2\alpha + 1)a_3 &= 3b_3 + \frac{2\alpha(1-\alpha)}{(\alpha+1)^2}b_2^2 + \beta \left(p_2 - \frac{1}{2}p_1^2\right) \\ &\quad + \frac{\beta^2(3\alpha+1)}{2(\alpha+1)^2}p_1^2 + \frac{2\beta(3\alpha+1)}{(\alpha+1)^2}p_1b_2. \end{aligned}$$

So, with

$$x = \frac{2(3\alpha+1) - 3(2\alpha+1)\mu}{(\alpha+1)^2},$$

we have

$$\begin{aligned} 3(2\alpha+1)(a_3 - \mu a_2^2) &= 3 \left(b_3 + \frac{1}{3}(x-2)b_2^2\right) \\ &\quad + \beta \left(p_2 + \frac{1}{4}(\beta x - 2)p_1^2\right) + \beta x p_1 b_2. \end{aligned} \quad (2.8)$$

Since rotations of  $f$  also belong to  $Q^\alpha(\beta)$ , without loss of generality, we may assume that  $a_3 - \mu a_2^2$  is positive. Thus we now estimate  $\operatorname{Re}(a_3 - \mu a_2^2)$ .

Since  $g \in \mathcal{C}$ , there exists  $h(z) = 1 + k_1z + k_2z^2 + \dots$  ( $|z| < 1$ ) with positive real part, such that  $g'(z) + zg''(z) = g'(z)h(z)$ . Hence, by equating coefficients, we get that  $b_2 = k_1/2$  and  $b_3 = (k_2 + k_1^2)/6$ . So, by Lemma 2.1,

$$\begin{aligned} 3\operatorname{Re}\left(b_3 + \frac{1}{3}(x-2)b_2^2\right) &= \frac{1}{2}\operatorname{Re}\left(k_2 - \frac{1}{2}k_1^2\right) + \frac{1}{4}(x+1)\operatorname{Re}k_1^2 \\ &\leq 1 - \rho^2 + (x+1)\rho^2 \cos 2\phi, \end{aligned} \quad (2.9)$$

where  $b_2 = k_1/2 = \rho e^{i\phi}$  for some  $\rho$  ( $0 \leq \rho \leq 1$ ). We also have

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$$\begin{aligned}\beta \operatorname{Re} \left( p_2 + \frac{1}{4}(\beta x - 2)p_1^2 \right) &= \beta \operatorname{Re} \left( p_2 - \frac{1}{2}p_1^2 \right) + \frac{1}{4}\beta^2 x \operatorname{Re} p_1^2 \\ &\leq 2\beta(1 - r^2) + \beta^2 x r^2 \cos 2\theta,\end{aligned}\quad (2.10)$$

where  $p_1 = 2re^{i\theta}$  for some  $r$  ( $0 \leq r \leq 1$ ). From (2.8-10), we obtain

$$\begin{aligned}3\operatorname{Re}(2\alpha + 1)(a_3 - \mu a_2^2) &\leq 1 - \rho^2 + (x + 1)\rho^2 \cos 2\phi \\ &\quad + 2\beta(1 - r^2) + \beta^2 x r^2 \cos 2\theta + 2\beta x r \rho \cos(\theta + \phi),\end{aligned}\quad (2.11)$$

and we now proceed to maximize the right-hand of (2.11). This function will be denoted by  $\psi(x)$  whenever all the parameters except  $x$  are held constant.

We consider first the case

$$\frac{2\alpha(1 - \alpha) + 2\beta(3\alpha + 1)}{3(2\alpha + 1)(1 + \beta)} \leq \mu \leq \frac{2(3\alpha + 1)}{3(2\alpha + 1)},$$

so that  $0 \leq x \leq 2/(1 + \beta)$ . Since the expression  $-2t^2 + t^2\beta x \cos 2\theta + 2xt$  is the largest when  $t = x/(2 - \beta x \cos 2\theta)$ , we have

$$-2t^2 + t^2\beta x \cos 2\theta + 2xt \leq \frac{x^2}{2 - \beta x \cos 2\theta} \leq \frac{x^2}{2 - \beta x}.$$

Thus

$$\begin{aligned}\psi(x) &\leq x + 1 + \beta \left( 2 + \frac{x^2}{2 - \beta x} \right) \\ &= 1 + 2\beta + \frac{2\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}{2(\alpha + 1)^2 - \beta\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}\end{aligned}$$

and with (2.11) this establishes the second inequality in the theorem. Equality occurs only if

$$p_1 = \frac{2x}{2 - \beta x} = \frac{2\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}{2(\alpha + 1)^2 - \beta\{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}, \quad p_2 = 2, \quad b_2 = b_3 = 1,$$

and the corresponding function  $f$  is defined by

$$(f'(z))^{1-\alpha}((zf'(z))')^\alpha = \frac{1}{(1-z)^2} \left( \lambda \frac{1+z}{1-z} + (1-\lambda) \frac{1-z}{1+z} \right)^\beta,$$



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$$\lambda = \frac{2(\alpha+1)^2 + (1-\beta)(2(3\alpha+1) - 3(2\alpha+1)\mu)}{4(\alpha+1)^2 - 2\beta(2(3\alpha+1) - 3(2\alpha+1)\mu)}.$$

We now prove the first inequality. Let

$$\mu \leq \frac{2\alpha(1-\alpha) + 2\beta(3\alpha+1)}{3(2\alpha+1)(1+\beta)},$$

so that  $x \geq 2/(1+\beta)$ . With  $x_0 = 2/(1+\beta)$ , we have

$$\begin{aligned} \psi(x) &= \psi(x_0) + (x - x_0)(\rho^2 \cos 2\phi + \beta^2 r^2 \cos 2\theta + 2\beta \rho r \cos(\theta + \phi)) \\ &\leq \psi(x_0) + (x - x_0)(1 + \beta)^2 \\ &\leq 1 + \frac{(1 + \beta)^2 \{2(3\alpha + 1) - 3(2\alpha + 1)\mu\}}{(\alpha + 1)^2} \end{aligned}$$

as required. Equality occurs only if  $p_1 = p_2 = 2$ ,  $b_2 = b_3 = 1$ , and the corresponding function  $f$  is defined by

$$(f'(z))^{1-\alpha}((zf'(z)))^\alpha = \frac{1}{(1-z)^2} \left( \frac{1+z}{1-z} \right)^\beta.$$

Let  $x_1 = -2/(1+\beta)$ . We shall find that  $\psi(x_1) = 1 + 2\beta$ , and the remaining inequalities follow easily from this one. By an argument similar to the one above, we obtain

$$\begin{aligned} \psi(x) &\leq \psi(x_1) + |x - x_1|(1 + \beta)^2 \\ &\leq -1 + \frac{(1 + \beta)^2 \{3(2\alpha + 1)\mu - 2(3\alpha + 1)\}}{(\alpha + 1)^2}. \end{aligned}$$

if  $x \leq x_1$ , that is,

$$\mu \geq \frac{2(\alpha+1)^2 + 2(3\alpha+1)(1+\beta)}{3(2\alpha+1)(1+\beta)}.$$

Equality occurs only if  $p_1 = 2i$ ,  $b_2 = i$ ,  $p_2 = -2$ ,  $b_3 = -1$ , and the corresponding function  $f$  is defined by

$$(f'(z))^{1-\alpha}((zf'(z)))^\alpha = \frac{1}{(1-iz)^2} \left( \frac{1+iz}{1-iz} \right)^\beta.$$

Also, for  $0 \leq \lambda \leq 1$ , we note that

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$$\begin{aligned}\psi(\lambda x_1) &= \lambda\psi(x_1) + (1 - \lambda)\psi(0) \\ &\leq \lambda(1 + 2\beta) + (1 - \lambda)(1 + 2\beta) = 1 + 2\beta,\end{aligned}$$

so  $\psi(x) \leq 1 + 2\beta$  for  $x_1 \leq x \leq 0$ , that is,

$$\frac{2(3\alpha + 1)}{3(2\alpha + 1)} \leq \mu \leq \frac{2(\alpha + 1)^2 + 2(3\alpha + 1)(1 + \beta)}{3(2\alpha + 1)(1 + \beta)}.$$

Equality occurs only if  $p_1 = b_2 = 0$ ,  $p_2 = 2$ ,  $b_3 = 1/3$ , and the corresponding function  $f$  is defined by

$$(f'(z))^{1-\alpha}((zf'(z))')^\alpha = \frac{1}{1-z^2} \left( \frac{1+z^2}{1-z^2} \right)^\beta = \frac{(1+z^2)^\beta}{(1-z^2)^{1+\beta}}.$$

We now show that  $\psi(x_1) \leq 1 + 2\beta$ . Since

$$\begin{aligned}&(-2 + \beta x_1 \cos 2\theta)t^2 + 2x_1 t \rho \cos(\theta + \phi) \\ &= (-2 + \beta x_1 \cos 2\theta) \left\{ t + \frac{x_1 \rho \cos(\theta + \phi)}{-2 + \beta x_1 \cos 2\theta} \right\}^2 + \frac{x_1^2 \rho^2 \cos^2(\theta + \phi)}{2 - \beta x_1 \cos 2\theta}\end{aligned}$$

for all real  $t$  and

$$2 - \beta x_1 \cos 2\theta = 2 + \frac{2\beta}{1 + \beta} \cos 2\theta \geq 2 - \frac{2\beta}{1 + \beta} \geq 0,$$

we have

$$\psi(x_1) - (1 + 2\beta) \leq \rho^2 \left( -1 + (x_1 + 1) \cos 2\phi + \frac{\beta x_1^2 (1 + \cos 2(\theta + \phi))}{2(2 - \beta x_1 \cos 2\theta)} \right).$$

Thus we consider the inequality

$$\beta x_1^2 (1 + \cos 2(\theta + \phi)) + 2(2 - \beta x_1 \cos 2\theta)(-1 + (x_1 + 1) \cos 2\phi) \leq 0.$$

After some simplifications, this becomes

$$4(\beta^2 (\cos 2\phi + 1)(\cos 2\phi - 1) - \beta(1 + \cos 2\theta + \sin 2\theta \sin 2\phi) - 1 - \cos 2\phi) \leq 0,$$

which is true if

$$2\beta^2 \cos^2 \theta \sin^2 \phi + 2\beta \cos \theta \sin \theta \cos \phi \sin \phi + \cos^2 \phi \geq 0. \quad (2.12)$$

Now, for all real  $t$ ,

$$2t^2 + 2t \sin \theta \cos \phi + \cos^2 \phi \geq 0,$$

so, by taking  $t = \beta \cos \theta \sin \phi$ , we obtain (2.12). This completes the proof of the theorem.

**Remark.** Letting  $\alpha = 0$  in Theorem 2.3, we have the corresponding result obtained by London [14], which extend the earlier results by several authors [1,5,10-12].

For  $\alpha = 1$  in Theorem, we have the following

**Corollary 2.1.** Let  $f \in Q^1(\beta)$  and be given by (1.1). Then for  $\beta \geq 0$ , we have

$$9|a_3 - \mu a_2^2| \leq \begin{cases} 1 + \frac{(1+\beta)^2(8-9\mu)}{4} & \text{if } \mu \leq \frac{8\beta}{9(1+\beta)}, \\ 1 + 2\beta + \frac{2(8-9\mu)}{8-\beta(8-9\mu)} & \text{if } \frac{8\beta}{9(1+\beta)} \leq \mu \leq \frac{8}{9}, \\ 1 + 2\beta & \text{if } \frac{8}{9} \leq \mu \leq \frac{8(2+\beta)}{9(1+\beta)}, \\ -1 + \frac{(1+\beta)^2(9\mu-8)}{4} & \text{if } \mu \geq \frac{8(2+\beta)}{9(1+\beta)}. \end{cases}$$

For each  $\mu$ , there is a function in  $Q^1(\beta)$  such that equality holds in all cases.

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